Transverse Combustion Instabilities: Modern Experimental Techniques and Analysis

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In this work we propose a new approach to elucidate specific combustion instability driving mechanisms and demonstrate application of the developed method to data collected using modern diagnostics. A laboratory-scale, optically accessible combustor was developed that exhibits full-scale-like, transverse combustion instabilities. We acquired simultaneous high-repetition rate (10 kHz) diagnostic data using particle image velocimetry, OH-planar laser induced fluorescence, CH\textsuperscript{*}-chemiluminescence, and dynamic pressure measurements. This paper discusses our approach to analyze the many gigabytes of data collected per second of testing. Specifically, we developed a technique based on the dynamic mode decomposition (DMD) and used it to extract interesting spatial and temporal features of the near-injector region from the simultaneously acquired data. This developed technique was used to analyze two datasets: one taken with the rig operating in a stable configuration and the other obtained when the rig was experiencing a transverse combustion instability. The analysis showed that both sets of data exhibited similar hydrodynamic and combustion modes/oscillations, suggesting that the combustor stability depends primarily on the phase of the of the energy transfer from the unsteady combustion heat release to the natural acoustic modes of the combustor.

Nomenclature

\begin{align*}
A & \quad \text{Finite dimensional approximation of the Koopman operator} \\
f & \quad \text{Acoustic frequency [Hz]} \\
I & \quad \text{Identity matrix} \\
L_1 & \quad \text{Left length of rig [m]} \\
L_2 & \quad \text{Right length of rig [m]} \\
\dot{m} & \quad \text{Mass flow rate [g/s]} \\
\tilde{S} & \quad \text{DMD matrix} \\
\Delta t & \quad \text{Time step [s]} \\
U & \quad \text{Left singular vector matrix} \\
v_k & \quad \text{Data snapshot at } k^{th} \text{ time step} \\
V & \quad \text{Data matrix/Krylov sequence} \\
W & \quad \text{Right singular vector matrix} \\
x & \quad \text{Transverse direction [mm]} \\
y & \quad \text{Injection direction [mm]} \\
\text{Im} & \quad \text{Imaginary part of complex number} \\
\text{Re} & \quad \text{Real part of complex number} \\
\mathcal{R}_j & \quad \text{Rayleigh index for SDD mode } j \\
\lambda & \quad \text{Complex Ritz value}
\end{align*}

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1. Introduction

Combustion instability (CI) due to feedback coupling between unsteady combustion heat release and one or more natural acoustic mode of the combustor can cause premature component failure in gas turbines and catastrophic failure in rocket engines. Predicting the onset of CI in practical devices requires understanding this feedback coupling, which involves interaction between the fluid flow, chemical reactions, and the acoustic field.\textsuperscript{1,2,3,4} We must study the interactions and feedback processes that drive CI to develop better predictive tools for guiding combustor design.

Experimental studies of CI feedback processes have proved to be challenging. Issues include the geometric dependence of experiments, testing cost, and limited diagnostic accessibility and capabilities, which restrict the information available for analysis. To address the first issue, experiments must be performed under full-scale conditions to properly simulate the acoustic field of the CI. This is due to the frequency dependence of the phenomena involved and because the acoustic frequencies in sub-scale rigs are higher than those in full-scale facilities. However, the cost of full-scale testing is prohibitively high. For example, much of the Apollo project’s estimated budget of $98 billion (2008 dollars)\textsuperscript{5} was for the development of the Rocketdyne F-1 liquid rocket engine (LRE), which was plagued by CIs. Additionally, most full-scale LRE tests have had limited diagnostics capabilities, consisting of a small number of dynamic pressure sensors and/or accelerometers. Such limited and restricted data can only yield the CI frequency and possibly the structure of the excited acoustic mode, but they do not provide details about the interaction of the acoustics with the fluid dynamic and combustion processes. These interactions are central to the driving feedback mechanism. In order to determine the feedback process interactions we must supplement the acoustic data with information from the fluid dynamic and combustion processes. To provide this additional information about the chemically reacting flow we apply modern, high-speed diagnostic techniques such as particle image velocimetry (PIV), to obtain detailed velocity fields, and planar laser induced fluorescence (PLIF), to obtain time-resolved reaction zone/combustion radical locations.

Thus, to practically and properly study the CI feedback process we need to adopt small-scale experimental approaches that can simulate the local conditions in a full-scale engine and provide optically access for modern diagnostics. In this study, we developed a small-scale combustor that exhibited full-scale like CIs simulating the environment within a larger combustor experiencing a first-tangential (1T) CI. This laboratory-scale rig maintains the proper transverse length scale of the large engine and, thus, maintains the proper transverse wave structure and frequency. In addition to being much smaller and less expensive to build and run, the developed small-scale rig is optically accessible, having a small quartz window, aligned with the axis of the injector for passing laser sheets into the combustor, and two larger windows for viewing the laser sheets and combustor volume. This allows us to take simultaneous, high-speed (10 kHz) PIV, OH-PLIF, CH\textsuperscript{*} -chemiluminescence, along with acoustic pressure measurements to explore the CI feedback process interactions.

These diagnostic techniques yield huge datasets, which have the potential to elucidate the fundamental processes that drive CIs, but the question remains as to how one would efficiently and effectively process these large datasets to find interactions between the fluid flow, chemistry, and acoustic processes. With
the rise in demand for processing large datasets, new techniques have been developed to extract useful information from these datasets. One of these new techniques, the dynamic mode decomposition (DMD), is based on the analysis of dynamic systems and can approximate some of the eigenvalues and eigenvectors of the system using measured data only. In this study, we developed a DMD based technique to analyze the measured high-speed data, where we scaled and combined different diagnostic data—from the same acquisition time—and processed with a standard DMD algorithm. This approach allowed us to find coupled modes showing the interaction between different physical phenomena, suggesting, e.g., that the velocity field can locally modify the OH radical intensity. Using this technique we analyzed data from two configurations of the laboratory-scale combustor: one obtained when the system was stable and the other obtained when the system exhibited an acoustically coupled CI. The results of the analysis are presented in this paper with the objective of determining the key similarities and differences between the stable and unstable configurations as the first step to improving our understanding of the interactions involved in CI feedback mechanisms.

2. Experiments

2.1. Experimental facility

To demonstrate how CI feedback mechanisms can be elucidated in a laboratory-scale LRE simulator by modern techniques, we first show how to perform these tests while maintaining the proper transverse wave structure and frequency. Figure 1a shows schematically how a small-scale rig could simulate the transverse acoustic environment of a full-scale engine. The light blue circle represents a transverse cross-section of a full-scale LRE experiencing a 1T CI whose acoustic pressure anti-nodes (+ and −) are located at the top and bottom of the cross-section and the direction of the acoustic velocity field inside the chamber is shown as arrows parallel to the acoustic motion. The red portion of Fig. 1a represents a rectangular arc section of the full-scale LRE that experiences acoustic velocity oscillations that are directed along its curved axis and has acoustic pressure anti-nodes at both ends. Assuming that the arc width is small compared with the chamber diameter, the acoustic oscillations along the axis would appear to be nearly one dimensional. Extracting this arc from the full-scale LRE and straightening it yields the configuration shown in Fig. 1b. This simplified configuration could serve as a small-scale rig for the study of a full-scale CI because it provides the proper orientation between the acoustic velocity and the mean flow directions while maintaining the correct
oscillatory frequency. This concept forms the basis for the small-scale rig that was developed and used in this study, see Fig. 2.

The developed combustors employs a single swirl-type injector that has a two path counter swirling air supply and annular fuel injection. The fuel used in this study was mixture of H$_2$ and CH$_4$ to obtain a highly reactive mixture and retain the ability to use CH$^*$-chemiluminescence. The mass flow rate of air, H$_2$, and CH$_4$ were nominally 11.5 g/s, 0.19 g/s, and 0.3 g/s, respectively.

![Rig concept and rig solid models](image)

Figure 2: Experimental facility details

Figure 3a shows the experimental facility, laser systems, and high-speed cameras. Figure 3b provides a close-up view of the main combustor section. Also shown in Fig. 3b are the window cooling systems and the exhaust ejector to protect optical equipment, e.g., the laser sheet forming optics, which, are located just above the combustor. This rig configuration and a second configuration based on active impedance boundary control are discussed in more detail by Quinlan and Zinn.

### 2.2. Experimental conditions

To better understand CIs we focus primarily on the differences between two experimental test conditions in this work. The conditions will be referred to as the stable and unstable cases. The only significant change in the test facility between the stable and unstable cases was the length of the left duct, which was $L_1 = 0.262 \text{ m}$ and $L_1 = 0.914 \text{ m}$ respectively (see Fig. 2b). The sensing section side of the transverse duct had a constant length of $L_2 = 1.016 \text{ m}$. Other operational parameters of interest are listed in Table 1. The Swirl percentage is defined by the mass flow rate of air through the outer, high-swirl number path relative to the total mass flow of air. The primary measured acoustic mode is one full-cycle along the major axis of the rig, and, to not confuse this with a tangential or longitudinal mode, this mode will be called the first widthwise mode (1W). The 1W frequency ($\tilde{f}$) is listed in Table 1, along with the magnitude of the power spectral density (PSD) of the acoustic pressure at $\tilde{f}$.

<table>
<thead>
<tr>
<th>Case</th>
<th>Total length</th>
<th>$\dot{m}_{\text{air}}$</th>
<th>$\dot{m}_{\text{H}_2}$</th>
<th>$\dot{m}_{\text{CH}_4}$</th>
<th>Swirl</th>
<th>$T_{\text{air}}$</th>
<th>$\tilde{f}$</th>
<th>PSD</th>
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<tr>
<td>stable</td>
<td>1.27</td>
<td>11.56</td>
<td>0.188</td>
<td>0.297</td>
<td>20.8</td>
<td>327.4</td>
<td>253.9</td>
<td>0.0035</td>
</tr>
<tr>
<td>unstable</td>
<td>1.93</td>
<td>11.58</td>
<td>0.189</td>
<td>0.313</td>
<td>21.1</td>
<td>316.6</td>
<td>169.7</td>
<td>0.0807</td>
</tr>
</tbody>
</table>
2.3. Diagnostic data

The specific high-speed diagnostics applied to the rig were: dynamic pressure sensing, PIV, OH-PLIF, and CH$^*$-chemiluminescence. A sample of the time history of the acoustic data is shown in Figs. 4a and 4b and the PSD in Figs. 4c and 4d for the stable and unstable datasets, respectively. The PSDs show that in both cases there is a large amount of spectral content in the 1 kHz range. The spectral peak is wide, which may be attributed to the flow noise of the air system.

An example of the instantaneous data collected by each of the optical techniques is shown in Fig. 5. Note that each of these images represents the same sampling period of the unstable dataset, hence they are simultaneous images. The axes scales are in mm, the injector nozzle is located at approximately $y = 0$ mm and has diameter 10.9 mm, and the injector axis/mean flow direction is from bottom to top.

The first image, Fig. 5a, shows the calculated PIV velocity field found from a Mie scattering image pair. Note, the velocity field has some areas where the vectors appear to be unusually large or small, evidence of the noise often encountered with PIV data. This noise is unavoidable and the data reduction approaches employed must be resilient to the presence of such noise. The instantaneous PLIF and chemiluminescence images are shown in Figs. 5b and 5c. Note that PIV and PLIF are planar diagnostics whereas chemiluminescence is a line of sight integrated technique.

We determined the temporal mean for each dataset, for each diagnostic, at each spatial location. These mean PIV and PLIF images for the unstable case dataset are shown in Figs. 6a and 6b. The first plot, Fig. 6a, shows the spatial distribution of the velocity vectors and the contour coloring represents the y-component of the velocity to highlight different regions of the flow field. The mean velocity data show a well defined inner recirculation zone, along with inner and outer shear layers of the swirling reacting flow. The inner recirculation zone is produced by vortex breakdown along the centerline of the flow. The locations of the shear layers are important because hydrodynamic instabilities of these shear layers grow into large scale vortical structures, which are responsible for large scale mixing. The second plot, Fig. 6b, shows the spatial dependence of the mean OH intensity. It shows that a conical reaction zone is stabilized below the central recirculation zone in the low velocity/stagnation region associated with the reverse flow. Near the injector, the reaction zone appears to be collocated with the inner shear layer. Notably, Figs. 6a and 6b
show that while the inner shear layer diverges slightly from the axis with downstream distance, the reaction zone divergence is significant. This situation is similar to that observed by Emerson et al. who studied bluff body stabilized flames and noted that when the low density region spread faster than the shear layer, there were strong hydrodynamic shear layer instabilities. This finding could provide a mechanism for nonlinear growth of perturbations near the anchor point as they propagate downstream. Figures 6a and 6b also show the velocity and combustion fields are asymmetric, especially the velocity field. Swirl stabilized flames are often asymmetric, especially in confined environments, due to the highly three dimensional structure of the...
3. Data analysis

The 10 kHz simultaneously acquired datasets discussed in the previous section are large in size, greater than 15 GB of data for a less than one second measurement period. Additionally, this data contains relatively high levels of noise due to the high-speed nature and the low energy per shot of the laser systems. From this data we want to extract as much useful engineering data, e.g., finding characteristic propagation velocities of disturbances for use in reduced-order modeling, and scientific understanding, e.g., how the aerodynamically held flame stabilization point moves with acoustic perturbations.

We will use the fact that the simultaneously acquired data provides different information about the same system at the same time, in part to reduce the noise due to random errors, but also to find coupled system-wide modes. In other words, modes that have coherent mono-frequency content in more than one set of measured diagnostic data, which represents different physical phenomena. To accomplish these goals, we have selected the DMD for the bulk of the data processing. The DMD method allows us to estimate some of the eigenvalues and eigenvectors of what is known as the Koopman operator, which is directly related to the governing equations through a measurement operator.\cite{6,11,12} Further, if the measurements that are made are close to the system state vector, then the DMD method can approximately recover the eigenvalues and eigenvectors of the governing equations—without knowledge of the governing equations.

3.1. DMD algorithm

The DMD method allows us to approximate some of the eigenvalues and eigenvectors of the governing equations by using only observations of the system. More specifically the DMD method allows us to find approximations to the eigenvalues and eigenvectors of the matrix $A$, which determines how the vector of measurements, also known as observations or snapshots, $v_k$, at time step $k$ evolves to the observation at the next time step, $k + 1$:

$$v_{k+1} = Av_k.$$ \hspace{1cm} (1)

Following the discussion of Schmid,\cite{6} snapshot vectors provided by experiments or numerical simulations at equally spaced time intervals may be arranged column-wise in a matrix:

$$V \equiv \{v_1 \ v_2 \ \cdots \ v_N\}.$$ \hspace{1cm} (2)

Using Eq. (1) the data matrix, $V$, may be rewritten in the following form:

$$V = \{v_1 \ A v_1 \ A^2 v_1 \ \cdots \ A^{N-1} v_1\}.$$ \hspace{1cm} (3)

From this expression we see that the data matrix also forms what is known as a Krylov sequence. Much is known about estimating eigenvalues from these sequences and many powerful mathematical tools are based on these concepts, namely the Arnoldi and Lanczos methods.\cite{13,14}
In the DMD method, we define $V_{1,N-1}$ and $V_{2,N}$ as:

$$V_{i,j} \equiv \{A^i v_1 \ A^j v_1 \ \cdots \ A^j v_1\}.$$  \hspace{1cm} (4)

By applying Eq. (1) to each column of $V_{1,N-1}$, we find that $V_{1,N-1}$ and $V_{2,N}$ are related by the matrix $A$:

$$AV_{1,N-1} = V_{2,N}.$$  \hspace{1cm} (5)

From Eq. (5), taking the SVD of $V_{1,N-1}$ (i.e., $V_{1,N-1} = U \Sigma W^H$) yields:

$$AU \Sigma W^H = V_{2,N}.$$  \hspace{1cm} (6)

Right multiplying Eq. (6) by $W$ and then $\Sigma^{-1}$ and left multiplying by $U^H$ gives:

$$U^H AU = U^H V_{2,N} W \Sigma^{-1} \equiv \tilde{S}.$$  \hspace{1cm} (7)

Key to the method, the matrix $\tilde{S}$ approximates a portion of the spectra of $A$ due to the semi-unitary character of $U$ and $W$ ($U^H U = W^H W = I$ but $UU^H \neq WW^H \neq I$) and $\tilde{S}$ may be obtained from the data matrix—without knowledge of $A$, i.e., without knowledge of the governing equations.

Experimentally obtained data contains noise and this is often especially true for high-speed diagnostic data. To address this, noise reduction is performed based on the SVD of the $m \times n$ matrix $V_{1,N-1}$ ($V_{1,N-1} = U \Sigma W^H$). To reduce the noise we seek a low-rank approximation to $V_{1,N-1}$ in some sense, here the least squares sense or equivalently in the Frobenius norm. \cite{15,16} The approximation’s rank, $r$, is less than $m$ and $n$ and is selected to balance the number of DMD modes we want to find and the amount of information rejected as noise. To find the approximation to $V_{1,N-1}$, we take the singular value matrix:

$$\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n),$$  \hspace{1cm} (8)

with

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0,$$  \hspace{1cm} (9)

and zero the singular values with index greater than $r$:

$$\tilde{\Sigma} = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r, 0, \ldots, 0).$$  \hspace{1cm} (10)

Therefore, the best approximation in the least squares sense to $V_{1,N-1}$ is:

$$\tilde{V}_{1,N-1} = U \tilde{\Sigma} W^H.$$  \hspace{1cm} (11)

Since some of the diagonal entries of $\tilde{\Sigma}$ are zero, the matrix is singular and hence non-invertible, but, we can find the Moore-Penrose pseudoinverse: \cite{17}

$$\tilde{\Sigma}^{-1} = \text{diag}(\sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_r^{-1}, 0, \ldots, 0).$$  \hspace{1cm} (12)

So, in Eq. (7) we use $\tilde{\Sigma}^{-1}$ rather than $\Sigma^{-1}$.

If the data snapshots have zero temporal mean, although not the case for our data, then the above process is equivalent to projecting the data into the first $r$ proper orthogonal decomposition (POD) modes. \cite{18}

The remaining step is to find the DMD modes from Eq. (7). To find the approximate eigenvalues and eigenvectors—commonly called Ritz values and Ritz vectors—from the matrix $\tilde{S}$ and how they are related to those of matrix $A$, we compare the two eigenproblems:

$$A \psi_i = \mu_i \psi_i,$$  \hspace{1cm} (13)

and:

$$\tilde{S} \phi_i = \mu_i \phi_i.$$  \hspace{1cm} (14)

We have already used the fact that for some $i$’s the eigenvalues in both equations are approximately the same. Left multiplying Eq. (13) by $U^H$ and letting $\psi_i = U \phi_i$ we see that:

$$U^H AU \phi_i = \mu U^H U \phi_i.$$  \hspace{1cm} (15)
Simplifying this expression yields Eq. (14), so the desired relationship is:

\[ \psi_i = U \phi_i. \]  

(16)

The eigenvalues can be transformed into the more typical form—with the real part representing the growth or decay parameter, telling us about the stability of the mode, and the imaginary part representing the angular frequency of the mode—given by the relationship:

\[ \lambda = \ln(\mu) / \Delta t. \]  

(17)

### 3.2. DMD analysis of PIV data

Initially, the DMD method was used to analyze the unstable PIV data to find the Ritz values and Ritz vectors (i.e., approximate eigenvalues and eigenvectors) of the velocity field. The resulting modes, an example of which is shown in Fig. 7, suggest the presence of an outer shear layer instability. Figures 7a and 7b show the real and imaginary parts of a complex Ritz vector, respectively. The second image may be thought of as the same mode shown in the first image, phase-shifted by 90°, which are conceptually analogous to the sine and cosine functions. This oscillatory pattern, spatially shifted between the two phases, shows the traveling wave nature of these modes. The contour coloring for these figures represent the x-direction component of the normalized velocity (each Ritz mode has been normalized to have an \( L_2 \) norm of 1) and the alternating bands of transverse velocity are indicative of vortical structures. The offset in phase of the vortical structure from the left to right half planes in Fig. 7 may be due to the presence of a helical vortex core wrapped around the inner recirculation zone, as shown by the POD mode analysis of Steinberg et al.\(^{19}\)

![Figure 7: Example of a DMD mode from the unstable experiment PIV (x-component) data](image)

### 3.3. DMD analysis of PLIF data

Next we used the DMD technique to analyze the unstable OH-PLIF dataset. One resulting mode is shown in Fig. 8. Again, a comparison of the mode’s real and imaginary parts suggests that this is a traveling wave mode. These modes show that the high OH intensity oscillations are primarily confined to a region near the outer edge of the time-average reaction zone, near the inner shear layer.

The above discussion suggests that applying the DMD method to the PIV and PLIF datasets separately, one may conclude that an outer shear layer instability drives combustion oscillations near the inner shear layer, which may not be the case. To illustrate the potentially unrelated nature of the outer shear layer instability and the combustion oscillations, we plotted, see Fig. 9, the Ritz values found in the separate analyses of the PIV and PLIF data for the unstable condition. In this figure the x-axis and y-axis represents the growth or decay parameter and the real angular frequency of the modes, respectively. Examination of Fig. 9 shows that the PIV (black circles) and PLIF (red squares) spectra significantly differ from one to another except for the Ritz value at the origin, which represents a non-dynamic steady mode. Since the two sets of Ritz values are considerably different, we may conclude that the outer shear layer instability and the
Figure 8: Example of an unstable PLIF dataset DMD mode

Figure 9: Ritz values for separate PIV (black circles) and PLIF (red squares) datasets

3.4. Simultaneous dynamic decomposition

This section presents a new approach that we have developed for analyzing the simultaneously acquired data, which we will refer to as the simultaneous dynamic decomposition (SDD) method. This effort was motivated by the inability of the previously used techniques to identify the salient dynamic features of the investigated CI mechanism from the separately processed data. Furthermore, since the CI feedback mechanism presumably depends on the fluid dynamics, combustion chemistry, and acoustic field, we had to find a way to process all of the measured data while maintaining the simultaneous nature of the data. This inclusive approach improved our results because the ability of the DMD algorithm to approximate the system dynamics is improved when the measurement vector more closely resembles the state vector.

3.4.1. The SDD concept

In this study, we developed a DMD based approach for analyzing simultaneously acquired, high-speed, diagnostic data. To demonstrate this concept, we present the method through an example utilizing the
simultaneously measured PIV and PLIF data discussed earlier. While not presented here, using this method, analyses have been preformed by additionally including the chemiluminescence and acoustic data.

Figure 10 shows a diagram that describes the manner in which the DMD method may be used to analyze simultaneously acquired data. While this concept was developed independently during this study, it is similar to the approach presented by Richecoeur et al.\textsuperscript{20} The SDD method begins, at each time step, by reshaping into column vectors and scaling (if necessary) the two dimensional data matrices (i.e., images) that represent the x-direction and y-direction velocity components and the OH-PLIF intensity. These vectors are then concatenated into a single, larger vector. Since the DMD method is invariant to row exchanges, the ordering of data within the vectors is arbitrary but must be consistent for all time steps. Recalling that the input vectors for the DMD are commonly referred to as \textit{snapshots}, the new vector containing the simultaneously measured data will be referred to as a \textit{super-snapshot}. Returning to the diagram, the super-snapshots from each time step become the input into the DMD algorithm, as presented in Section 3.1.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{sdd_diagram.png}
\caption{The SDD conceptual data flow}
\end{figure}

Also from Section 3.1, we recall that the input to the DMD algorithm was a series of snapshots equally spaced in time and the output was a set of complex Ritz values, each of which is associated with a Ritz vector. The Ritz values describe the modal angular frequency and growth or decay rate and the Ritz vectors represent the spatial dependence of the mode. In the SDD algorithm, the Ritz vectors calculated by the DMD method contain information from both the velocity and OH intensity snapshots. In fact, the vectors have the same component-wise structure as the input super-snapshots, i.e., if the \( j^{th} \) component of the super-snapshots corresponded with the y-direction velocity component at location \((13 \text{ mm}, 47 \text{ mm})\), then the \( j^{th} \) component of the Ritz vectors corresponds to the (normalized) y-direction velocity component at location \((13 \text{ mm}, 47 \text{ mm})\) for each mode. As a result, for each Ritz value, the Ritz vectors may be broken back into smaller vectors and reshaped into images corresponding to the coupled x-direction velocity component, y-direction velocity component, and OH-PLIF intensity.

Conceptually, if we use the DMD method as discussed in Sections 3.2 and 3.3—which analyzed each diagnostic dataset separately—then the Ritz values and Ritz vectors we find depend only on the analyzed dataset, as demonstrated by Eq. (18):

\[
\begin{align*}
\mathbf{v}_{\text{PIV}} & \rightarrow \text{DMD} \rightarrow \lambda_{\text{PIV}}(\mathbf{v}_{\text{PIV}}), \ \phi_{\text{PIV}}(\mathbf{v}_{\text{PIV}}) \\
\mathbf{v}_{\text{PLIF}} & \rightarrow \text{DMD} \rightarrow \lambda_{\text{PLIF}}(\mathbf{v}_{\text{PLIF}}), \ \phi_{\text{PLIF}}(\mathbf{v}_{\text{PLIF}}).
\end{align*}
\]

On the other hand, the SDD method finds a single set of Ritz values that is dependent on both sets of diagnostic data:

\[
\begin{equation}
\mathbf{v} = \begin{bmatrix} \mathbf{v}_{\text{PIV}} \\ \mathbf{v}_{\text{PLIF}} \end{bmatrix} \rightarrow \text{DMD} \rightarrow \lambda(\mathbf{v}_{\text{PIV}}, \mathbf{v}_{\text{PLIF}}), \ \phi(\mathbf{v}_{\text{PIV}}, \mathbf{v}_{\text{PLIF}}) = \begin{bmatrix} \phi_{\text{PIV}}(\mathbf{v}_{\text{PIV}}, \mathbf{v}_{\text{PLIF}}) \\ \phi_{\text{PLIF}}(\mathbf{v}_{\text{PIV}}, \mathbf{v}_{\text{PLIF}}) \end{bmatrix}.
\end{equation}
\]

Equation (19) shows that the Ritz modes \( \phi_{\text{PIV}} \) and \( \phi_{\text{PLIF}} \) are now dependent on both the PIV and PLIF data. In other words, the modes found by the SDD method, \( \phi \), represent the coupled oscillation of both the
velocity, $\phi_{PIV}$, and OH intensity, $\phi_{PLIF}$, at the complex frequency $\lambda$. Showing two things, this method of processing simultaneously measured data preserves the spatial and temporal relationship between the multiple diagnostic measurements and locates modes with common dynamics ($\lambda$) in the physical phenomena represented by the data.

### 3.4.2. SDD analysis of PIV and PLIF data

To illustrate the advantages of the SDD method, we will use it to analyze the simultaneously measured, high-speed, PIV and PLIF data from the stable and unstable tests.

#### 3.4.2.1. Typical results

We present SDD results for the unstable test data and discuss the results in detail to show what type of information may be accessed by this approach, then the stable and unstable experimental measurements will be compared in the next section.

Figure 11 shows data obtained from the unstable case and contains plots of the normalized singular value distributions, a Ritz value map, and representative Ritz vectors. Figure 11a presents the normalized singular values, plotted on a log-log scale. The x-axis represents the index of the singular value and the y-axis the singular value magnitude normalized by the largest singular value. In this example, the relative singular value threshold used was equal to $0.03$ for forming the pseudoinverse. Any singular values below this threshold are regarded as noise. The analysis showed that there were 21 singular values that exceeded this threshold value. As a result, the DMD algorithm produces 21 Ritz values and Ritz eigenvectors. These complex Ritz values were converted from the exponential form ($\mu$) to the more typical time-stepper form ($\lambda$) by Eq. (17) and were plotted on the complex plane in Fig. 11b where the x-axis is the growth or decay factor and the y-axis is the real angular frequency. Three points are highlighted on this plot: the diamond at the origin, labeled c-d, represents the steady mode, see Figs. 11c and 11d, which is approximately equivalent to the (scaled) temporal mean; the circled Ritz value, labeled e-f, shows the Ritz value associated with the Ritz mode displayed in Figs. 11e and 11f; and the square box, labeled g-h, is associated to the mode in Figs. 11g and 11h. Each oscillatory image pair represents the real normalized velocity and real normalized OH intensity of a single Ritz mode—associated with a single Ritz value. Since the real and imaginary portions of the oscillatory modes are similar in structure and represent a $\pm \pi/2$ phase shift in the mode, we will typically show only the real or imaginary part of a mode since the physical interpretation of the two parts is similar. The first pair of images, shown in Figs. 11c and 11d, describe the spatial distribution of the steady mode. The contours represent the y-direction velocity component to highlight the central recirculation zone and the steady OH intensity image employs a false colormap to highlight the time-average reaction zone. In Figs. 11e to 11h the red-blue color in the left plots represents positive-negative x-velocities, respectively, to highlight vortical structures. The red-blue color in the right plots represents more-less OH intensity than the steady mode.

The first oscillatory mode, shown in Figs. 11e and 11f, has a frequency of 1460 Hz and both the velocity and OH intensity exhibit similar spatial structures; i.e., this mode represents a coupled oscillation in the velocity and OH intensity at 1460 Hz. The structure has a wavelength, shown in Fig. 11e, of approximately 30 mm. Thus, using the wavelength and frequency it is possible to estimate a convective velocity for this structure, here approximately $36 \text{ m/s}$. Turning to the second oscillatory mode, Figs. 11g and 11h, at 1837 Hz, a similar structure is found but with a shorter characteristic wavelength, approximately 20 mm. Again the convective velocity is approximately equal to $36 \text{ m/s}$. For these modes and others the calculations yielded similar convective velocities. This sequence of similar structures with increasing frequency and decreasing size, which progressively have their largest amplitudes closer to the injector, is a characteristic of related DMD modes found in a parabolic arc like series of values on the eigenvalue map. Two such parabolas are shown by dashed lines in Fig. 11b. Furthermore, in both oscillatory modes there is a positive correlation between the x-velocity and increased OH intensity on the right of the injector centerline and a negative correlation on the left. The change in sign of the correlation is due to the change in the direction of the velocity with respect to the flame propagation direction on the opposite sides of the injector axis. We interpret this finding physically as the x-direction velocity component of vortical structures alternately push the reaction zone in on itself—decreasing OH intensity locally—and push the reaction zone into fresh reactants—locally enhancing combustion. The reaction zone augmentation by the vortical structures is shown schematically in Fig. 12.
Figure 11: Typical SDD data (unstable configuration dataset)
Another use for the SDD is to take the initially noisy data, decompose into the contributions from each SDD mode and reconstruct the images from these contributions. In other words, with the SDD modes the original, noisy data at any time, $v_k$, may be projected into those modes—finding the SDD modal coefficient—and then the snapshot may be reconstructed or synthesized. Essentially filtering the data through the SDD modes. As an example, we project a raw (noisy) instantaneous velocity field (shown in Fig. 13a) and OH intensity field into the 21 SDD modes found above. We use the coefficients from this projection to reconstruct the instantaneous velocity and OH fields. These fields have been de-noised by the process as we can see in the velocity field shown in Fig. 13b.

![Figure 12: Hydrodynamic and combustion interaction](image)

![Figure 13: Noise reduction and coherent mode dynamic highlighting with DMD synthesis](image)

Some of the features in the raw image appear to be missing from the reconstructed image, but the reconstructed image only retains the coherent mono-frequency dynamic structures found within the 21 SDD modes. More dynamical modes and more noise would be retained if more singular values were retained when forming the pseudoinverse. We have added streamlines to Fig. 13b to highlight the tight vortical structure wrapped around the inner recirculation zone, which is hard to identify in the more complex velocity field of Fig. 13a. This process of projection and synthesis allows us to identify the important dynamical features more easily than possible from the original noisy data.

### 3.4.2.2. Stable vs. unstable experimental datasets

To identify key similarities and differences between the stable and unstable combustor configurations described in Section 2.2, the PIV and PLIF data from two experiments were analyzed with the SDD method. Both datasets were processed using the SDD method described in Section 3.4.1, with a normalized singular value threshold of 0.03, which yielded 25 modes in the stable case and 21 modes in the unstable case. Figure 14 shows the combined PIV and PLIF spectra for the stable (red squares) and unstable (black circles) datasets. Inspecting this figure we find that a large number of Ritz values in one spectra have a corresponding mode in the second spectra that has a similar Ritz value. We did not expect this similarity because of the considerably difference in stability between the two cases. Furthermore, this similarity may indicate
that some of the key fluid dynamics and combustion processes are largely the same in both the stable and unstable cases.

Figure 14: Ritz values from the SDD method, stable (red squares) and unstable (black circles) experiments

Figure 15 shows a selection of oscillatory modes from the stable and unstable combustor configurations taken from the upper parabolic arc like structure found in Fig. 14. The mode shown in Figs. 15a and 15b has a similar Ritz value as the mode shown in Figs. 15e and 15f and so on. Comparing these figures we find that not only are the Ritz values similar, but the mode shapes are also similar. This correspondence between modes further supports the conclusion that the hydrodynamic and combustion processes are similar in the stable and unstable cases. So, if we can identify and model the differences between these two cases, in the future, it may be possible to use data acquired from stable experiments to predict unstable operation.

To this point we have not identified how the high-frequency hydrodynamic modes and induced combustion oscillations can provide energy to lower frequency acoustic modes. To see how this transfer might occur we continue the discussion started in Section 3.4.2.1 about how the modes along an arc in the Ritz value complex plane have related structures with high-frequency/short wavelength structures closer to the injector and low-frequency/long wavelength structures further downstream. Physically, as a coherent structure propagates downstream (away from the injector) it grows in size due to diffusion/viscous effects and the pairing of two or more vortical structures into a larger structure. This growth in structure size and decrease in frequency is likely a key link between the higher frequency hydrodynamic oscillations and the lower frequency, transverse acoustic modes. For example, if the coherent structures grow in size to the point where their characteristic frequency is close to that of an acoustic natural frequency, the oscillatory motion of the coherent structure would make the heat release fluctuate, which acts as a source of energy for the natural acoustic mode.

Analyzing the unstable experimental dataset, we found SDD modes with frequencies of 101 Hz and 221 Hz, which are near the CI frequency of 170 Hz. The 221 Hz mode is shown in Fig. 16. While not directly related to the 170 Hz acoustic mode, the SDD mode structure oscillates in the x or transverse direction and is similar to the structure found in the velocity data PSD at 170 Hz (see Fig. 17a). To create Fig. 17 we analyze the data at each spatial location and calculate the PSD from the time history and plotted is the difference of the power spectrum at 170.9 Hz and the noise floor at 183.1 Hz. In this figure the contours represent the PSD content of the x-direction velocity components and the OH intensity at each spatial location for the unstable combustor data.

Linear theory predicts that there will be no transfer of energy between modes at different frequency, but the system is nonlinear and energy transfer between modes of differing frequencies can occur when the frequency difference is not too large. Conceptually this is similar to the energy transfer by nonlinear terms in the turbulent energy cascade. So the transverse structures and frequencies of the SDD mode and the 170 Hz PSD might indicate that it is possible for nonlinear transfer of energy between the hydrodynamics, combustion, and acoustics.
Figure 15: SDD modes from stable and unstable configurations

(a) Stable: Velocity at 329 Hz
(b) Stable: OH at 329 Hz
(c) Stable: Velocity at 2029 Hz
(d) Stable: OH at 2029 Hz
(e) Unstable: Velocity at 454 Hz
(f) Unstable: OH at 454 Hz
(g) Unstable: Velocity at 1837 Hz
(h) Unstable: OH at 1837 Hz
The above discussed observations and the previously discussed combustion zone structure, see Fig. 6c, suggest a feedback process that may be responsible for the observed CI. Recalling from Fig. 17b that there were 170 Hz oscillations in OH intensity throughout the reaction zone, but that the fluctuations were larger in amplitude further downstream. These larger OH intensity oscillations are not likely due to direct perturbation from the transverse acoustic field because the magnitude of the fluid dynamic velocities in this region may be on the order of 50 m/s, while the transverse acoustic velocities are orders of magnitude smaller. However, near the stagnation/reaction zone anchor point, the acoustic velocity is large compared to the local fluid dynamic velocities. Hence, this important region of the flow field may be directly influenced by the acoustic waves.

We begin our description of the proposed CI feedback mechanism with the perturbation of the reaction zone anchor point. First, the stagnation point of the aerodynamically stabilized reaction zone is perturbed by the transverse acoustic velocity field. This perturbation of the inner recirculation zone may then excite hydrodynamic waves of the inner shear layer that convect—at an approximate constant velocity—downstream. As they convect downstream, these waves grow into large scale structures, which perturb the reaction zone. The processes of acoustic perturbation of the reaction zone anchoring and coherent structure propagation are shown schematically in Fig. 18. The perturbation of the unstable shear layer results in a nonlinear growth through which the small hydrodynamic waves become large enough to affect the combustion process, although initially at the high frequencies of the hydrodynamic instability. The growth and pairing processes then cause the frequency to drop to the point where energy may be transferred nonlinearly to the transverse acoustic modes. The combustion heat release oscillations cause an unsteady volumetric expansion, which acts as an acoustic source. If some of the new acoustic energy goes into the 170 Hz acoustic mode (at the correct phase) then the feedback cycle is complete.

Since the SDD analyses revealed that many of the modes were similar when the combustor was operating
in the stable and unstable configurations, it is necessary to understand why CI was not excited in the stable case. Recalling that in the stable configuration the combustor was shorter and its natural acoustic mode frequencies larger, it is possible that in this case the heat release oscillations could not couple with the natural acoustic modes of the combustor. Likely, due to an incompatibility of the oscillatory combustion frequencies/phases and the natural acoustic modes as dictated by the sign of the Rayleigh index (positive: driving, negative: damping):

$$R_j = \int_{\text{cycle}} \dot{q}_j \dot{p}_j \, dt.$$  \hspace{1cm} (20)

Also, because the combustor geometry was different between the two cases, the location of the combustion relative to the standing acoustic velocity and pressure fields are different. The change in pressure field directly influences the Rayleigh index and the acoustic velocity oscillations likely effect the oscillatory combustion—again changing the Rayleigh index and the stability.

4. Conclusions and future work

We showed that analyzing the PIV and PLIF datasets separately with the DMD method yielded important dynamical content in both diagnostics—such as high-frequency OH intensity fluctuations and an outer shear layer instability. But, the modes from each of the analyses are allowed to have independent dynamical information, i.e., the modes found in each analysis can have unrelated frequencies. Since the frequencies, in general, differ from the PIV to PLIF analyses, we have no way to determine if the modes are related.

This suggested that we analyze the data in such a way as to maintain the simultaneous nature of the data. This was accomplished by forming composite input vectors for the DMD algorithm to ensure that a dynamic mode output from the algorithm had the same frequency and growth factor for the both the PIV and PLIF data. Some of these modes, in both the stable and unstable operation cases, showed what appears to be an inner shear layer instability and closely related combustion zone oscillations. We found these modes had a nearly constant convective velocity. Furthermore, the SDD method showed the hydrodynamics and combustion were relatively unchanged regardless of the system’s stability; suggesting that the change in the relative phases of the combustion with the acoustic velocity/pressure waves and acoustic frequency—both changes due to the change of the combustor’s transverse length—have the largest effect on the process of passing energy from the unsteady heat release to the acoustic modes and the system’s stability. The central role of the phase shift is speculation at this point and will be further examined in future work. If we can establish this link between the combustion and the acoustics, it may be possible to create a general framework—including making measurements of a combustor operating in a stable manner and modeling the acoustic feedback mechanism—to predict and prevent detrimental CIs.
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